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Density sets of sets of positive integers ☆

Georges Grekos [a], Ladislav Mišík [b,*], János T. Tóth [c]

[a] Université Jean Monnet (Saint-Etienne), Mathématiques, 23 rue du Dr. Paul Michelon, 42023 Saint-Etienne Cedex 2, France
[b] University of Ostrava, Department of Mathematics, 30. Dubna 22, 701 03 Ostrava 1, Czech Republic
[c] Department of Mathematics University J. Selye, PO Box 54, 94501 Komárno, Slovak Republic

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ABSTRACT

Extending previous results, we give a new description of the density set, that is the set of all pairs of densities – upper and lower – of all subsets of a given set of positive integers. The extension consists in using the concept of weighted density with the weight function satisfying two standard conditions. In order to prove that the density set is convex, we establish and use the joint Darboux property of the weighted density. Finally we prove that the density set is closed through an explicit characterization of its upper boundary.

1. Introduction

Throughout the paper we will use the following notation

$$\mathbb{N} = \{1, 2, 3, \ldots\}, \qquad \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\} \quad \text{and} \quad \overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}.$$

Solutions of many problems in number theory need some information on size of particular sets of positive integers. Asymptotic and logarithmic densities provide one of the most important ways how to do it. Both these kind of densities can be generalized by concept of weighted densities introduced in [1] (see also [6]) and defined as follows. Let $f : \mathbb{N} \to [0, \infty)$ be a weight function with $f(1) > 0$. For a set $A \subset \mathbb{N}$ and a number $n \in \mathbb{N}$ let us denote

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* Corresponding author.
E-mail addresses: grekos@univ-st-etienne.fr (G. Grekos), ladislav.misik@osu.cz (L. Mišík), toth.janos@selyeuni.sk, janos.toth@osu.cz (J.T. Tóth).

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$$A_f(n) = \sum_{a \in A,\ a \leqslant n} f(a)$$

and define

$$\underline{d}_f(A) = \liminf_{n\to\infty} \frac{A_f(n)}{\mathbb{N}_f(n)} \quad \text{and} \quad \bar{d}_f(A) = \limsup_{n\to\infty} \frac{A_f(n)}{\mathbb{N}_f(n)},$$

the lower and the upper f-density of the set A, respectively. If a common value $d_f(A) = \underline{d}_f(A) = \bar{d}_f(A)$ exists, we simply speak about f-density. The above mentioned asymptotic and logarithmic densities are obtained by special choices of weight functions $f_0(n) = 1$ and $f_1(n) = \frac{1}{n}$ for all $n \in \mathbb{N}$, respectively. In the sequel we will suppose that the weight function f fulfils the following conditions

$$\sum_{n=1}^{\infty} f(n) = \infty \tag{D}$$

and

$$\lim_{n\to\infty} \frac{f(n)}{\mathbb{N}_f(n)} = 0. \tag{L}$$

Although densities provide important information on size, in general they say nothing about distribution of elements of the set. For example consider the set $X = \bigcup_{n=1}^{\infty} (2^{2n}, 2^{2n+1}] \cap \mathbb{N}$. It consists of big blocks and gaps and it is easy to see that the lower and upper asymptotic densities of X are $\frac{1}{3}$ and $\frac{2}{3}$, respectively. On the other hand, there exist sets with the same asymptotic densities as X, but containing no big block and gaps. To go inside the distribution structure of sets of positive integers one needs some other, more sensitive means.

One of them is the so-called f-density set of the set $A \subset \mathbb{N}$ defined as $S_f(A) = \{(\bar{d}_f(X), \underline{d}_f(X));\ X \subset A\}$. In [2,3] some general properties of $S_f(A)$ are established for functions f fulfilling (D) and

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n-1} |f(k+1) - f(k)|}{\mathbb{N}_f(n)} = 0. \tag{S}$$

In this case it is proved that $S_f(A)$ is a closed convex subset of the trapezium $PQRS$ and it contains the triangle with vertices PQR, where $P = (0,0)$, $Q = (\bar{d}_f(A), 0)$, $R = (\bar{d}_f(A), \underline{d}_f(A))$, $S = (\underline{d}_f(A), \underline{d}_f(A))$. Thus, to find a complete description of $S_f(A)$, it is sufficient to determine the function $b_A : [0, \bar{d}_f(A)] \to [0, \underline{d}_f(A)]$ whose graph forms the upper boundary of $S_f(A)$.

In the paper [4] relations of density set to the gap structure of A are studied and some general properties of the function b_A are found. Nevertheless, the problem of determining b_A explicitly in terms of A and f fulfilling (D) and (S) remained open. The purpose of this paper is to determine the f-density set of the set A completely in terms of A and f for f fulfilling (D) and (L). Notice that the condition (L) is less restrictive than (S), thus no results from the above cited papers can be applied.

2. Results

We will start with two propositions.

Proposition 1. *Let $A \subset \mathbb{N}$ and $\theta \in [0,1]$. Then there is a set $B \subset A$ such that $\underline{d}_f(B) = \theta \underline{d}_f(A)$ and $\bar{d}_f(B) = \theta \bar{d}_f(A)$.*

Proof. If $\theta = 0$ or $\theta = 1$, then $B = \emptyset$ or $B = A$ respectively, trivially fulfil the statement. Thus suppose $0 < \theta < 1$. We can also suppose that $\bar{d}_f(A) > 0$ (otherwise take $B = A$). Define B as follows

$$B = \{n \in A;\ n \geqslant 2 \text{ and } B_f(n-1) + f(n) \leqslant \theta A_f(n)\}.$$

Thus $1 \notin B$ and every $n \geqslant 2$ belongs to B if and only if both conditions

$$n \in A \tag{1}$$

and

$$B_f(n-1) + f(n) \leqslant \theta A_f(n) \tag{2}$$

are satisfied. As $B_f(n) = B_f(n-1) + \chi_B(n) f(n) \leqslant B_f(n-1) + f(n)$ (here χ_B stands for the characteristic function of the set B), we can simply deduce from (2) that the condition

$$B_f(n) \leqslant \theta A_f(n) \tag{3}$$

holds for every $n \in \mathbb{N}$.

Now we will show that the set $A \setminus B$ is infinite. Suppose the contrary, i.e. that there is a positive integer c such that $B_f(n) = A_f(n) - c$ for all large values of n. It would follow from (3) that $A_f(n) - c \leqslant \theta A_f(n)$, that is $A_f(n) \leqslant \frac{c}{1-\theta}$ for every large n. The last inequality yields $d_f(A) = 0$, a contradiction to our hypothesis.

For n sufficiently large, let

$$m = m(n) = \max((A \setminus B) \cap [1, n]).$$

For any such m we have

$$B_f(m) \leqslant \theta A_f(m) < B_f(m) + f(m).$$

Thus for n sufficiently large, such that $m = m(n)$ exists, we have

$$\begin{aligned} 0 &\leqslant \theta A_f(n) - B_f(n) = \theta A_f(m) - B_f(m) + \theta(A_f(n) - A_f(m)) - (B_f(n) - B_f(m)) \\ &= \theta A_f(m) - B_f(m) - (1-\theta)(A_f(n) - A_f(m)) \\ &\leqslant \theta A_f(m) - B_f(m) < f(m). \end{aligned}$$

Dividing by $\mathbb{N}_f(n)$ we obtain

$$0 \leqslant \theta \frac{A_f(n)}{\mathbb{N}_f(n)} - \frac{B_f(n)}{\mathbb{N}_f(n)} < \frac{f(m)}{\mathbb{N}_f(n)} \leqslant \frac{f(m)}{\mathbb{N}_f(m)}.$$

Since the last term tends to 0 as n tends to ∞, it follows that the quantities $\theta \frac{A_f(n)}{\mathbb{N}_f(n)}$ and $\frac{B_f(n)}{\mathbb{N}_f(n)}$ have the same limits inferior and superior and the statement of the proposition follows. $\square$

The previous proposition says that f-density has the so-called *Darboux property* (see e.g. [5, p. 217]).

Definition 1.

(i) Let d be a density $d : \mathcal{D} \to [0,1]$ defined on a family $\mathcal{D}$ of subsets of $\mathbb{N}$. We say that d has the Darboux property if and only if for any $A \in \mathcal{D}$ and $\theta \in [0,1]$ there is a $B \subset A$ such that $B \in \mathcal{D}$ and $d(B) = \theta d(A)$.

(ii) Let d be a density notion to which there are associated an upper density $\bar{d} : \mathcal{D} \to [0,1]$ and a lower density $\underline{d} : \mathcal{D} \to [0,1]$, both defined on the same family $\mathcal{D}$.

We say that d has the joint Darboux property if and only if for any $A \in \mathcal{D}$ and $\theta \in [0,1]$ there is a $B \subset A$ such that $B \in \mathcal{D}$, $\bar{d}(B) = \theta \bar{d}(A)$ and $\underline{d}(B) = \theta \underline{d}(A)$.

Corollary 1. *Let $f : \mathbb{N} \to [0,\infty)$ be a weight function fulfilling condition* (D)*. Then the f-density has the joint Darboux property with $\mathcal{D} = \mathcal{P}(\mathbb{N})$ if and only if the function f fulfils the condition* (L)*.*

Proof. If (L) is satisfied, then it is proved in Proposition 1 that the f-density has the joint Darboux property. Now suppose that (L) is not satisfied. This means that there is a $\delta > 0$ such that

$$\limsup_{n\to\infty} \frac{f(n)}{\mathbb{N}_f(n)} = \delta.$$

It follows that there is a set $N = \{n_1 < n_2 < \cdots\} \subset \mathbb{N}$ such that $\lim_{j\to\infty} \frac{f(n_j)}{\mathbb{N}_f(n_j)} = \delta$. Clearly

$$\delta \leqslant \underline{d}_f(N) \leqslant \bar{d}_f(N) \leqslant 1.$$

Let $\theta = \frac{\delta}{2\underline{d}_f(N)}$. Then no set $B \subset N$ satisfies $\underline{d}_f(B) = \theta \underline{d}_f(N) = \frac{\delta}{2}$ because either B is infinite and consequently $\underline{d}_f(B) \geqslant \delta$ or B is finite and then, by (D), $\underline{d}_f(B) = 0$. $\square$

Proposition 2. *Let $A \subset \mathbb{N}$ and $b \in [0, \underline{d}_f(A)]$. Then there is a set $B \subset A$ such that $\bar{d}_f(B) = \bar{d}_f(A)$ and $\underline{d}_f(B) = b$.*

Proof. If $b = \underline{d}_f(A)$ then $B = A$ trivially fulfils the statement. Thus suppose that $0 \leqslant b < \underline{d}_f(A)$. Put $\theta = \frac{b}{\underline{d}_f(A)}$. Let A' be the set the existence of which was proved in Proposition 1. That is

$$A' \subset A, \quad \underline{d}_f(A') = \theta \underline{d}_f(A) = b, \qquad \bar{d}_f(A') = \theta \bar{d}_f(A).$$

Let $0 < p_1 < q_1 < p_2 < q_2 < \cdots$ denote integers (to be determined). The required set B will be of the form

$$B = \left(\bigcup_{n=1}^{\infty} \big((p_n, q_n] \cap A' \big) \right) \cup \left(\bigcup_{n=1}^{\infty} \big((q_n, p_{n+1}] \cap A \big) \right).$$

Thus $A' \subset B \subset A$ which implies that

$$b = \underline{d}_f(A') \leqslant \underline{d}_f(B) \quad \text{and} \quad \bar{d}_f(B) \leqslant \bar{d}_f(A).$$

Now, in order to obtain equalities in the above two inequalities, it is sufficient to choose each integer q_n (resp. p_{n+1}) sufficiently large with respect to p_n (resp. q_n). For instance, since the set

$(B \cap [1, p_n]) \cup (A' \cap (p_n, \infty))$ has lower f-density equal to b, one can choose q_n such that

$$\left| \frac{B_f(q_n)}{\mathbb{N}_f(q_n)} - b \right| < \frac{1}{n}.$$

Similarly we can choose p_{n+1} such that

$$\left| \frac{B_f(p_{n+1})}{\mathbb{N}_f(p_{n+1})} - \overline{d}_f(A) \right| < \frac{1}{n}$$

and the statement of proposition follows. $\square$

By Proposition 2, the upper boundary of $S_f(A)$ provides substantial information for the density set. We are going to determine it. Suppose that $A = \{a_1, a_2, a_3, \ldots\} \subset \mathbb{N}$ where $a_1 < a_2 < a_3 < \cdots$. Let $x \in [0, \overline{d}_f(A)]$, $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Define

$$g_f(A, x, n, k) = \frac{x\mathbb{N}_f(a_n) + (A_f(a_{n+k}) - A_f(a_n))}{\mathbb{N}_f(a_{n+k})},$$
$$g_f(A, x, n) = \inf\{g_f(A, x, n, k);\ k \in \mathbb{N}_0\}$$

and

$$g_f(A, x) = \liminf_{n\to\infty} g_f(A, x, n).$$

We will prove that the graph of the function $g_f(A, x)$ forms the upper bound of $S_f(A)$. Before doing so, let us establish some properties of the function $g_f(A, x)$.

Proposition 3. *For every $A \subset \mathbb{N}$ the function $g_f(A, x)$ has the following properties.*

(a) *The function $g_f(A, x)$ is continuous and nondecreasing with respect to x. Moreover, for all $x, y \in [0, \overline{d}_f(A)]$ the inequality $|g_f(A, x) - g_f(A, y)| \leqslant |x - y|$ holds.*
(b) *The inequality $g_f(A, x) \leqslant \min\{x, \underline{d}_f(A)\}$ holds for all numbers $x \in [0, \overline{d}_f(A)]$ and moreover $g_f(A, \overline{d}_f(A)) = \underline{d}_f(A)$.*

Proof. (a) The fact that the function $x \mapsto g_f(A, x)$ is nondecreasing is a straightforward consequence of the definition of $g_f(A, x)$. Let $x, y \in [0, \overline{d}_f(A)]$. Then for all $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$

$$\left| \frac{x\mathbb{N}_f(a_n) + (A_f(a_{n+k}) - A_f(a_n))}{\mathbb{N}_f(a_{n+k})} - \frac{y\mathbb{N}_f(a_n) + (A_f(a_{n+k}) - A_f(a_n))}{\mathbb{N}_f(a_{n+k})} \right|$$
$$= |x - y| \frac{\mathbb{N}_f(a_n)}{\mathbb{N}_f(a_{n+k})} \leqslant |x - y|.$$

Thus $|g_f(A, x, n) - g_f(A, y, n)| \leqslant |x - y|$ holds for all $n \in \mathbb{N}$ which yields the required inequality and also the continuity of $g_f(A, x)$ follows.

(b) Let $x \in [0, \overline{d}_f(A)]$ and $n \in \mathbb{N}$. Then

$$x = \frac{x\mathbb{N}_f(a_n) + (A_f(a_{n+0}) - A_f(a_n))}{\mathbb{N}_f(a_{n+0})} = g_f(A, x, n, 0) \geqslant g_f(A, x, n)$$

which yields the first part of the inequality. To prove its second part, notice that

$$g_f(A,x,n) \leqslant \liminf_{k\to\infty} \frac{x\mathbb{N}_f(a_n) + (A_f(a_{n+k}) - A_f(a_n))}{\mathbb{N}_f(a_{n+k})} = \underline{d}_f(A).$$

Applying the just proved inequality, to prove the last equality, it is sufficient to show $g_f(A,\bar{d}_f(A)) \geqslant \underline{d}_f(A)$. Let us suppose the contrary, i.e. $g_f(A,\bar{d}_f(A)) < \underline{d}_f(A)$. Then there exist a positive number $v < \underline{d}_f(A)$, a sequence of positive integers $n_1 < n_2 < \cdots$ and a sequence (k_i) in $\mathbb{N}_0$ such that

$$g_f\bigl(A,\bar{d}_f(A),n_i,k_i\bigr) = \frac{\bar{d}_f(A)\mathbb{N}_f(a_{n_i}) + (A_f(a_{n_i+k_i}) - A_f(a_{n_i}))}{\mathbb{N}_f(a_{n_i+k_i})} < v. \tag{4}$$

Let $\varepsilon > 0$. Then there is $i_\varepsilon \in \mathbb{N}$ such that for all $i > i_\varepsilon$ we have $\frac{A_f(a_{n_i})}{\mathbb{N}_f(a_{n_i})} < \bar{d}_f(A) + \varepsilon$. Applying (4) and the last inequality, we get

$$\begin{aligned}\frac{A_f(a_{n_i+k_i})}{\mathbb{N}_f(a_{n_i+k_i})} &= \frac{A_f(a_{n_i}) + (A_f(a_{n_i+k_i}) - A_f(a_{n_i}))}{\mathbb{N}_f(a_{n_i+k_i})}\\ &< \frac{(\bar{d}_f(A)+\varepsilon)\mathbb{N}_f(a_{n_i}) + (A_f(a_{n_i+k_i}) - A_f(a_{n_i}))}{\mathbb{N}_f(a_{n_i+k_i})} < v + \varepsilon.\end{aligned}$$

Thus $\underline{d}_f(A) \leqslant v < \underline{d}_f(A)$, a contradiction. $\square$

Theorem 1. *Let $A \subset \mathbb{N}$. Then $S_f(A) = \{(x,y);\ x \in [0,\bar{d}_f(A)],\ y \in [0,g_f(A,x)]\}$.*

Proof. It is sufficient to prove

(i) $\underline{d}_f(B) \leqslant g_f(A,\bar{d}_f(B))$ for every $B \subset A$,
(ii) for every $x \in [0,\bar{d}_f(A)]$ there exists a set $B \subset A$ such that $\bar{d}_f(B) = x$ and $\underline{d}_f(B) = g_f(A,x)$,
(iii) for every $x \in [0,\bar{d}_f(A)]$ and every $y \in [0,g_f(A,x))$ there exists a set $C \subset A$ such that $\bar{d}_f(C) = x$ and $\underline{d}_f(C) = y$.

Proof of (i). Let $B \subset A$, $\varepsilon > 0$ and $n_0 \in \mathbb{N}$. Then there is a $n_\varepsilon \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n > n_\varepsilon$, it is

$$\frac{B_f(n)}{\mathbb{N}_f(n)} < \bar{d}_f(B) + \varepsilon. \tag{5}$$

On the other hand, by the definition of $g_f(A,\bar{d}_f(B))$, there exist integers $m_\varepsilon > \max\{n_\varepsilon,n_0\}$ and $k_\varepsilon \in \mathbb{N}_0$ such that

$$\frac{\bar{d}_f(B)\mathbb{N}_f(a_{m_\varepsilon}) + (A_f(a_{m_\varepsilon+k_\varepsilon}) - A_f(a_{m_\varepsilon}))}{\mathbb{N}_f(a_{m_\varepsilon+k_\varepsilon})} < g_f\bigl(A,\bar{d}_f(B)\bigr) + \varepsilon. \tag{6}$$

Then, taking into account that $a_{m_\varepsilon} > n_\varepsilon$, by (5) we have

$$\frac{B_f(a_{m_\varepsilon})}{\mathbb{N}_f(a_{m_\varepsilon})} < \bar{d}_f(B) + \varepsilon. \tag{7}$$

Using (6) and (7), we have

$$\begin{aligned}
\frac{B_f(a_{m_\varepsilon+k_\varepsilon})}{\mathbb{N}_f(a_{m_\varepsilon+k_\varepsilon})} &= \frac{B_f(a_{m_\varepsilon}) + (B_f(a_{m_\varepsilon+k_\varepsilon}) - B_f(a_{m_\varepsilon}))}{\mathbb{N}_f(a_{m_\varepsilon+k_\varepsilon})} \\
&< \frac{(\overline{d}_f(B)+\varepsilon)\mathbb{N}_f(a_{m_\varepsilon}) + (A_f(a_{m_\varepsilon+k_\varepsilon}) - A_f(a_{m_\varepsilon}))}{\mathbb{N}_f(a_{m_\varepsilon+k_\varepsilon})} \\
&= \frac{\overline{d}_f(B)\mathbb{N}_f(a_{m_\varepsilon}) + (A_f(a_{m_\varepsilon+k_\varepsilon}) - A_f(a_{m_\varepsilon}))}{\mathbb{N}_f(a_{m_\varepsilon+k_\varepsilon})} + \varepsilon\frac{\mathbb{N}_f(a_{m_\varepsilon})}{\mathbb{N}_f(a_{m_\varepsilon+k_\varepsilon})} \\
&< g_f\big(A, \overline{d}_f(B)\big) + \varepsilon + \varepsilon = g_f\big(A, \overline{d}_f(B)\big) + 2\varepsilon.
\end{aligned}$$

We get that for every $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ there is $n = m_\varepsilon + k_\varepsilon > n_0$ with $\frac{B_f(n)}{\mathbb{N}_f(n)} < g_f(A, \overline{d}_f(A)) + 2\varepsilon$. Thus $\underline{d}_f(B) \leqslant g_f(A, \overline{d}_f(B))$.

Proof of (ii). Let $x \in [0, \overline{d}_f(A)]$. Define $B \subset A$ by induction as follows

$$a_1 \notin B \tag{8}$$

and, supposing that $B \cap \{a_1, a_2, \dots, a_n\}$ has already been defined,

$$a_{n+1} \in B \quad \text{if and only if} \quad \frac{B_f(a_n) + f(a_{n+1})}{\mathbb{N}_f(a_{n+1})} \leqslant x. \tag{9}$$

First we will show that $\overline{d}_f(B) = x$. Notice that, by definition of B, the inequality $\frac{B_f(a_n)}{\mathbb{N}_f(a_n)} \leqslant x$ holds for every $n \in \mathbb{N}$, so

$$\overline{d}_f(B) = \limsup_{n\to\infty} \frac{B_f(a_n)}{\mathbb{N}_f(a_n)} \leqslant x. \tag{10}$$

On the other hand, suppose that $\overline{d}_f(B) < x$. Then there exists an $\varepsilon > 0$ such that the inequality $\frac{B_f(a_n)}{\mathbb{N}_f(a_n)} < x - \varepsilon$ holds for all sufficiently large values of $n \in \mathbb{N}$. Then, by the definition of B and taking into account (L), the set $A - B$ is finite. Consequently, $\overline{d}_f(B) = \overline{d}_f(A) \geqslant x$, a contradiction. Hence $\overline{d}_f(B) = x$.

Now, taking into account the result (i), to finish the proof it is sufficient to establish

$$\underline{d}_f(B) \geqslant g_f(A, x). \tag{11}$$

By Proposition 3, (11) holds in the case when $A - B$ is finite. So, let $A - B$ be infinite and suppose the contrary to (11), i.e. $\underline{d}_f(B) < g_f(A, x)$. Then there exists a positive real number $v < g_f(A, x)$ and an infinite sequence of positive integers $(n_p)_{p\in\mathbb{N}}$ such that

$$\frac{B_f(a_{n_p})}{\mathbb{N}_f(a_{n_p})} < v. \tag{12}$$

For every positive integer $p > 1$, let m_p denote the greatest integer less than n_p such that $a_{m_p} \notin B$. Notice that $\lim_{p\to\infty} a_{m_p} = \infty$ since $A - B$ is infinite. By the definition of B we have

$$\frac{B_f(a_{m_p})}{\mathbb{N}_f(a_{m_p})} > x$$

and, starting from (12), for all $p \in \mathbb{N}$ we get

$$\begin{aligned} v > \frac{B_f(a_{n_p})}{\mathbb{N}_f(a_{n_p})} &= \frac{B_f(a_{m_p}) + (B_f(a_{n_p}) - B_f(a_{m_p}))}{\mathbb{N}_f(a_{m_p+(n_p-m_p)})} \\ &> \frac{x\mathbb{N}_f(a_{m_p}) + (A_f(a_{n_p}) - A_f(a_{m_p}))}{\mathbb{N}_f(a_{m_p+(n_p-m_p)})} \\ &\geqslant g_f(A, x, a_{m_p}) \geqslant g_f(A, x) > v, \end{aligned}$$

a contradiction.

Proof of (iii). Having already proved (i) and (ii), (iii) follows from Proposition 2. □

The next theorem generalizes the already mentioned results in [2,3]. For the purpose of its proof, let us slightly reformulate the definition of $g_f(A,x,n)$. Define $g_f(A,x,n,\infty) = \underline{d}_f(A)$ for all $x \in [0, \overline{d}_f(A)]$ and $n \in \mathbb{N}$. Notice that the original definition of $g_f(A,x,n)$ is now equivalent to the following one

$$g_f(A,x,n) = \min\{g_f(A,x,n,k);\ k \in \overline{\mathbb{N}}_0\}.$$

Further, for $x \in [0, \overline{d}_f(A)]$ and $n \in \mathbb{N}$ denote by $k_n(x)$ the least element in $\overline{\mathbb{N}}_0$ such that

$$g_(A,x,n) = g_f\big(A,x,n,k_n(x)\big). \tag{13}$$

Also recall the notation $P = (0,0)$, $Q = (\overline{d}_f(A), 0)$, $R = (\overline{d}_f(A), \underline{d}_f(A))$, $S = (\underline{d}_f(A), \underline{d}_f(A))$.

Theorem 2. *Let the weight function f fulfil conditions* (D) *and* (L) *and $A \subset \mathbb{N}$. Then the f-density set $S_f(A)$ is closed and convex subset of the trapezium PQRS containing the triangle PQR.*

Proof. Closedness of $S_f(A)$ follows from Theorem 1. The fact that it is contained in the trapezium *PQRS* follows directly from definitions and basic properties of the lower and upper f-densities. Evidently, $P \in S_f(A)$ as $\emptyset \subset A$, $R \in S_f(A)$ as $A \subset A$ and $Q \in S_f(A)$ by Proposition 1. Thus, the fact that $S_f(A)$ contains the triangle *PQR* follows from convexity of $S_f(A)$ which we are going to prove.

By Theorem 1, to prove the convexity of $S_f(A)$, it is sufficient to prove the concavity of its upper boundary, the function $g_f(A,x)$ with respect to the variable x, i.e. to prove that

$$g_f(A, x_\alpha) \geqslant \alpha g_f(A, x_0) + (1-\alpha) g_f(A, x_1)$$

holds for all $0 \leqslant x_0 \leqslant x_1 \leqslant \overline{d}_f(A)$ and $\alpha \in [0,1]$, where $x_\alpha = \alpha x_0 + (1-\alpha)x_1$. So, let x_0, x_α, x_1 be as above. A straightforward calculation proves that for every $n \in \mathbb{N}$ and $k \in \overline{\mathbb{N}}_0$

$$g_f(A, x_\alpha, n, k) = \alpha g_f(A, x_0, n, k) + (1-\alpha) g_f(A, x_1, n, k). \tag{14}$$

Using (13) and (14) we have

$$\begin{aligned} g_f(A, x_\alpha, n) &= g_f\big(A, x_\alpha, n, k_n(x_\alpha)\big) = \alpha g_f\big(A, x_0, n, k_n(x_\alpha)\big) + (1-\alpha) g_f\big(A, x_1, n, k_n(x_\alpha)\big) \\ &\geqslant \alpha g_f(A, x_0, n) + (1-\alpha) g_f(A, x_1, n) \end{aligned}$$

which proves the concavity of $g_f(A, x, n)$ for all $n \in \mathbb{N}$. Finally,

$$\begin{aligned} g_f(A, x_\alpha) &= \liminf_{n\to\infty} g_f(A, x_\alpha, n) \\ &\geqslant \liminf_{n\to\infty}\bigl(\alpha g_f(A, x_0, n) + (1-\alpha) g_f(A, x_1, n)\bigr) \\ &\geqslant \liminf_{n\to\infty} \alpha g_f(A, x_0, n) + \liminf_{n\to\infty} (1-\alpha) g_f(A, x_1, n) \\ &= \alpha g_f(A, x_0) + (1-\alpha) g_f(A, x_1) \end{aligned}$$

and the concavity of $g_f(A, x)$ with respect to x follows. $\square$

References

[1] R. Alexander, Density and multiplicative structure of sets of positive integers, Acta Arith. 12 (1967) 321–332.
[2] G. Grekos, Sur la répartition des densités des sous-suites d'une suite donnée, PhD thesis, Université Pierre et Marie Curie, 1976.
[3] G. Grekos, Répartition des densités des sous-suites d'une suite d'entiers, J. Number Theory 10 (1978) 177–191.
[4] G. Grekos, B. Volkmann, On densities and gaps, J. Number Theory 26 (1987) 129–148.
[5] M. Mačaj, L. Mišík, T. Šalát, J. Tomanová, On a class of densities of sets of positive integers, Acta Math. Univ. Comenian. 72 (2003) 213–221.
[6] Š Porubský, Notes on density and multiplicative structure of sets of generalized integers, in: Topics in Classical Number Theory, Colloq. Budapest 1981, vol. II, in: Colloq. Math. Soc. J. Bolyai, vol. 34, 1984, pp. 1295–1315.